3.8 Complex Zeros and The Fundamental Theorem of Algebra

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DEFINITION: A **complex polynomial function** is one whose coefficients are complex numbers.

Recall that the set of **Real** numbers are a subset of the set of **Complex** numbers. For this reason, both polynomials below are examples of complex polynomials.

(1) $f(x) = 2x^3 - \sqrt{7}x^2 + (2i)x - 3$ and

(2) $g(x) = 2x^3 - 3x^2 + 1$

Recall that some polynomials may have roots that are **real, complex,** or both. Let's see…

(3) Find all of the zeros of $f(x) = x^3 - x^2 + 9x - 9$.

1, -3i, 3i

(4) Rewrite the cubic polynomial above as a product of linear factors of the form $f(x) = a(x - z_1)(x - z_2)(x - z_3)$ where z_i are complex zeros.

__(x-1)(x+3i)(x-3i)______

The above prepares you to understand the most important theorem of algebra:

<u>THE FUNDAMENTAL THEOREM OF ALGEBRA</u>

Every complex polynomial P(x) of degree $n \geq 1$ has exactly **n** zeros (provided a zero of multiplicity **m** is counted as **m** zeros), and can be factored into **n** linear factors of the form

$$p(x) = a(x - z_1)(x - z_2)(x - z_3)\ldots(x - z_n)$$

(5) How many roots must the equation $x^4 + 3x^2 - 4 = 0$ have? Express the polynomial as a product of linear factors.

4, (x-1)(x+1)(x-2i)(x+2i)

As you noticed, some of the roots of a polynomial come in pairs, as the next theorem indicate.

CONJUGATE PAIRS THEOREM(*) - If P(x) is a polynomial with **real** coefficients, and $a + bi$ is an imaginary root of the equation P(x) = 0, then $a - bi$ is also a root.
(*) ***same is true for irrational zeros if p(x) is a polynomial with rational coefficents … if $a+\sqrt{b}$ is a zero, then $a-\sqrt{b}$ is also a zero.***

6) A cubic equation with real coefficients has roots -2 and $3 + 5i$. What is the third root? 3-5i

7) Can a cubic polynomial have 3 zeros that are imaginary numbers? Explain.
__No, for the 3rd root to be imaginary its conjugate would have to be a root, but that would make a fourth root.__________

Your conclusion in #7 prepares you to understand this last theorem:

THEOREM: A polynomial of odd degree with real coefficients has at least one real zero.

8) A quartic equation with real coefficients has roots $2i$ ***and*** $3i$. Find the other 2 roots and write the equation as a product of 2 quadratic factors.

-2i, -3i, $(x^2+4)(x^2+9)$___

OBTAINING A QUADRATIC EQUATION FROM ITS ROOTS

A quadratic equation that has as roots the numbers z_1 *and* z_2, may be obtained by using the formula:

$$x^2 - (\text{sum of the roots})x + (\text{product of the roots}) = 0$$

9) Find a quadratic equation with roots $3 \pm 2i$.

$x^2 - 6x + 13 = 0$

10) Find a cubic polynomial that has zeros -5 *and* $2 + 7i$.

$(x+5)(x^2-4x+53)=x^3-4x^2+53x+5x^2-20x+265= x^3+x^2+33x+265$

10) Find a polynomial f of degree 5 and real coefficients that has zeros:
$3, -5i,$ *and* $-2 + 4i$.

$(x-3)(x^2+25)(x^2+4x+20)=x^5+x^4+33x^3-35x^2+200x-1500$

$f(x) =$ ______________________