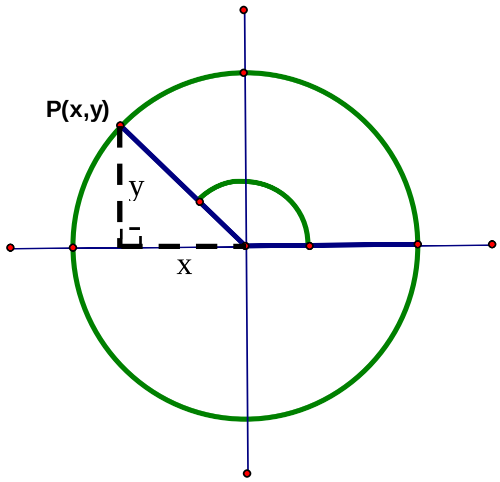
5.2 <u>THE SINE AND COSINE FUNCTIONS AND THE UNIT CIRCLE</u> (PART I)

By Mrs. Trebat for Pre-Calculus as edited by Mr. Benjamin

Suppose $P(x,y)$ is a point on the circle $x^2 + y^2 = r^2$ and θ is an angle in standard position with terminal ray $\overrightarrow{OP}$.



We define the **sine** of θ, denoted by $\sin\theta$, and the **cosine** of θ, $\cos\theta$ by:

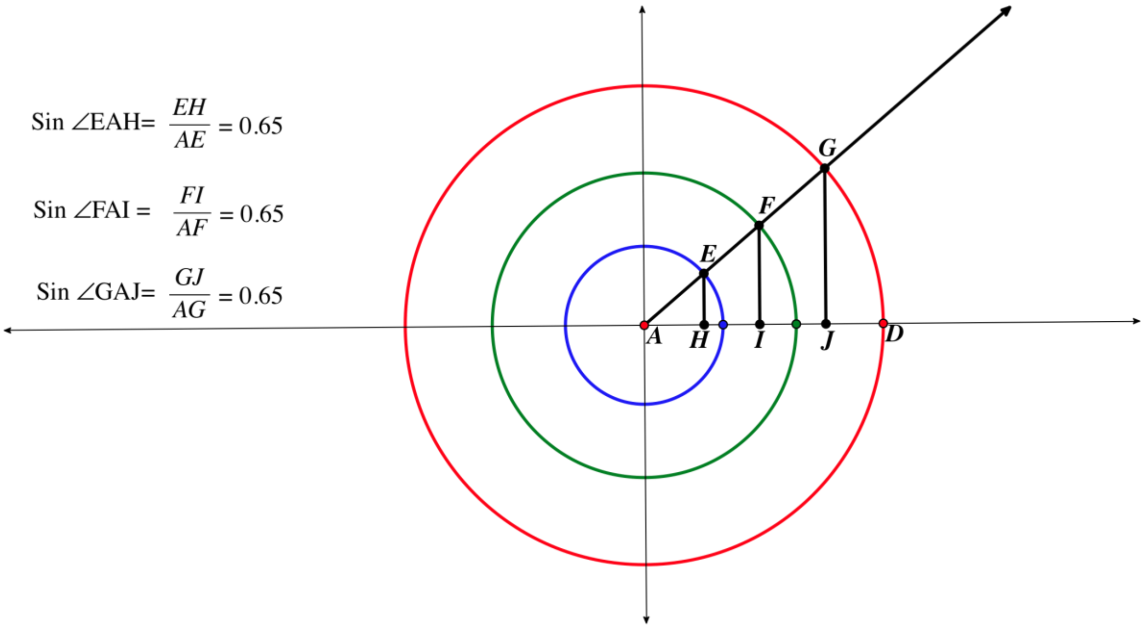
$$\sin\theta = \frac{y}{r} \quad \text{and} \quad \cos\theta = \frac{x}{r}$$

Example:

1) If the terminal ray of an angle θ in standard position passes through (-3,4), find $\sin\theta$ and $\cos\theta$. Make a sketch.

2) If θ is an angle in the fourth quadrant and $\sin\theta = -\frac{5}{13}$, find $\cos\theta$.

Important: Although the definitions of sin θ and cos θ involve the radius ***r*** of a circle, the **values of sin θ and cos θ depend only on θ as shown below**.



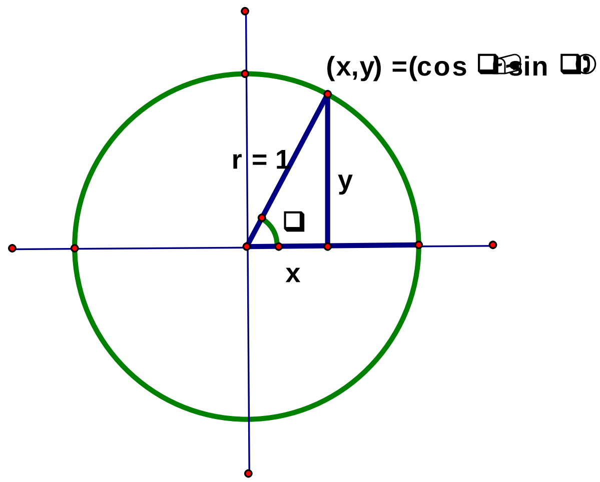
Since the value of r is irrelevant as shown above, it is easier to use the UNIT CIRCLE to define and study the trigonometric functions.   These functions are also called ***circular functions*** precisely because the circle is used in their definition. They are not limited to the sine and cosine functions as we'll see later…

THE UNIT CIRCLE

The circle defined by $x^2 + y^2 = 1$ has radius 1 which is why it is called the **unit circle.** The radius of 1 simplifies calculations. We additionally simplify our work by using the familiar angles of the 30-60-90 triangle.

Since the radius of this circle is 1, all points on the unit circle that lie on the terminal ray of θ have **x** and y coordinates equal to the cosine and sine of θ.

$$\boxed{\sin\theta = \frac{y}{r} = \frac{y}{1} = y \quad \text{and} \quad \cos\theta = \frac{x}{r} = \frac{x}{1} = x}$$



Complete the table with the values of the sine and cosine of the quadrantal angles 0°, 90°, 180°, 270°, and 360°. You will need this as a referece.

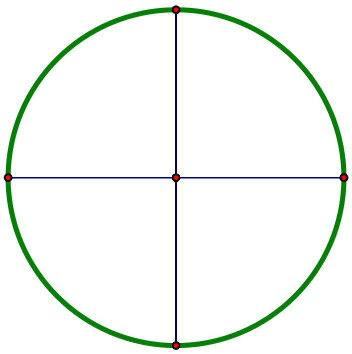
QUADRANTAL ANGLES

3)

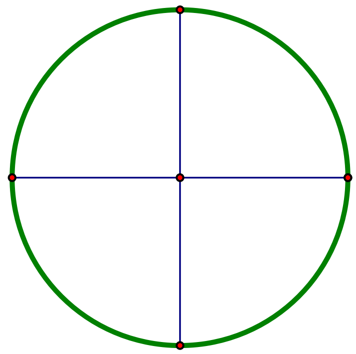
θ (degrees)	θ (radians)	$\sin\theta$	$\cos\theta$
0°			
90°			
180°			
270°			
360°			

4)

a) For which quadrants will the **sine** be positive? Negative? Complete the diagram on the left with a + or - to indicate whether the sine is positive or negative. Do the same for the cosine using the diagram on the right.



$\sin\theta = y$



$\cos\theta = x$

b) Describe what happens to the values of sin θ as angle θ increases from 0° to 90°. ______________________________________

c) Same as above for cos θ. ________________________________

5) Name each quadrant described:

a) $\sin\theta > 0$ ***and*** $\cos\theta < 0$ ________

b) $\sin\theta < 0$ ***and*** $\cos\theta < 0$ ________

c) $\sin\theta < 0$ ***and*** $\cos\theta > 0$ ________

6) Use the unit circle to justify the fact that for all θ:

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

7) Solve $\sin\theta = 1$ for **all** possible values of θ in degrees and radians.

PERIODICITY OF SINE AND COSINE FUNCTIONS

Since the sine and cosine repeat their values every 360° or 2π radians we say that the sine and cosine are **PERIODIC** functions. We will see later how they can be used to describe phenomena that are repetitive such as tides, sound waves, and more.

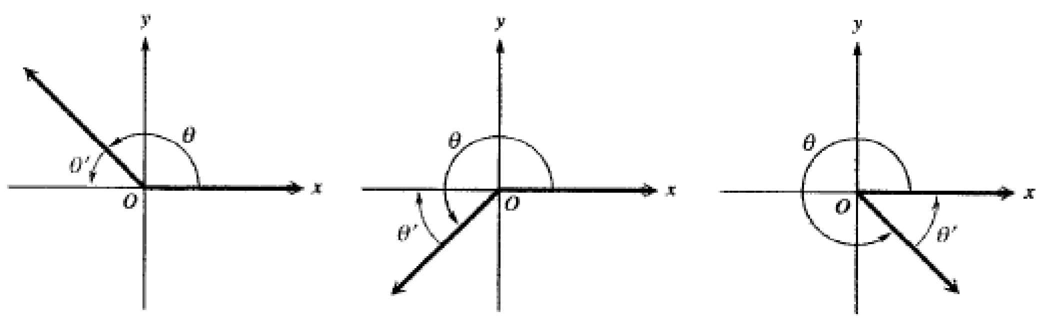
We summarize the fact that the sine and cosine functions have period $360°$ or 2π by writing:

$\sin(\theta + 360°) = \sin\theta$	$\cos(\theta + 360°) = \cos\theta$
$\sin(\theta + 2\pi) = \sin\theta$	$\cos(\theta + 2\pi) = \cos\theta$

Ready for Problem Set 1: (1-16)

REFERENCE ANGLES

DEFINITION: A **reference angle** for an angle θ, written θ', is the positive acute angle formed by the terminal side of angle θ and the x-axis as shown below.

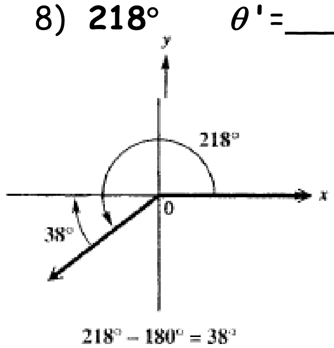


θ in quadrant II $\qquad$ θ in quadrant III $\qquad$ θ in quadrant IV

WATCH OUT! *The reference angle is always found with reference to the* ***x-axis,*** *never the y-axis!*

Find the reference angles for the following angles. Draw a sketch.

8) **218°** $\quad \theta'$=____ $\qquad\qquad$ 9) **160°** $\quad \theta'$=____



10) **315°** $\quad \theta'$=____ $\qquad\qquad$ 11) **695°** $\quad \theta'$=____

12) Could an angle of measure $95°$ be a reference angle? Explain.

____

13) What is the measure of the angle in Quad III that uses $\theta' = \frac{\pi}{3}$ as its reference angle? ________

USING YOUR CALCULATOR TO DRAW CONCLUSIONS ABOUT AN ANGLE AND ITS REFERENCE ANGLE...(Set MODE to degrees)

14) a) What is $\sin 12°$? ______
b) The angle $\theta = 168°$ is an angle in Quadrant II that uses $\theta' = 12°$ as its reference angle. What is $\sin 168°$? _____

15) a) What is cos 35°? ______
b) Which angle θ in quad IV uses $\theta' = 35°$ as its reference angle? ______. What is $\cos\theta$? _____
c) What is $\cos 215°$? How do you compare this with cos 35°? Explain.
____

16) a) What is cos 23°? ______
b) How many different sines and cosines for angles less than 360° can you name based on knowing the cos23°? (Hint: I got 7 others)

17) Find the $\sin 30°$ and use this answer to figure out the sine of the following angles without using the calculator:
a) $\sin 150°$ ____ b) $\sin 210°$ ____ c) $\sin 330°$ ____

The examples above suggest that to find the trig function values of any angle θ, **use the values of its reference angle θ' with the appropriate signs.**

Let's use this approach to find the exact values of the sine and cosine of angles that use 30°,45°, and 60° as their reference.

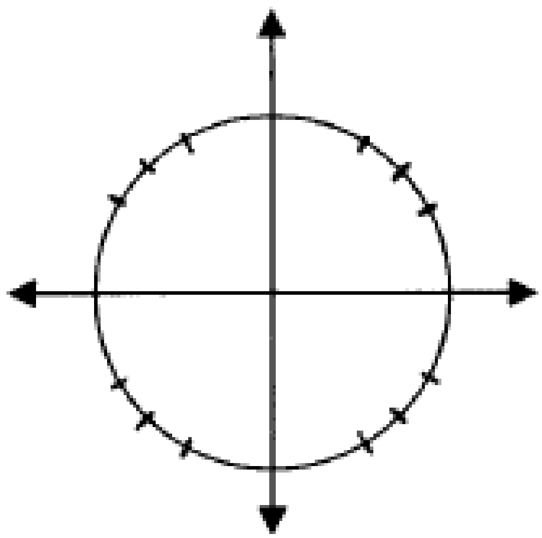
FINDING SINE AND COSINE OF SPECIAL ANGLES

The diagram below shows a unit circle with marks that correspond to angles of 30°, 45° and 60° in quadrant I. The angles in the other quadrants use these 3 angles as their reference angles.

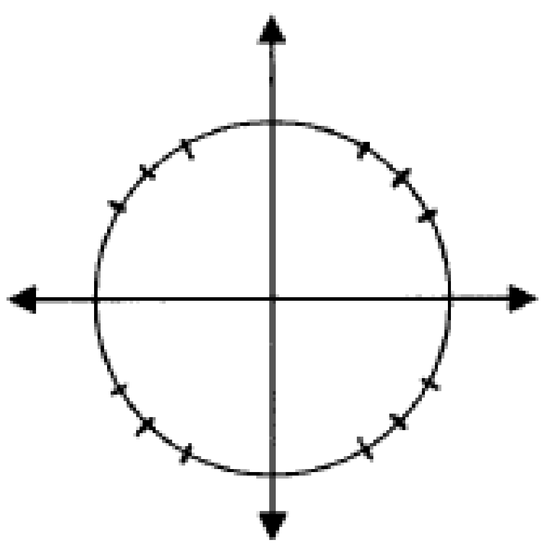
17) **Angles with reference angle $45°$ or $\frac{\pi}{4}$:**

a) Draw all four rays corresponding to the 45° reference angle.
b) From the point on the terminal side of the 45° ray, drop a perpendicular to the x-axis, forming a right triangle.
c) Use the 45°-45°-90° triangle ratios to find the sine and cosine of the 45° angle. Complete:
$x = \cos 45° =$ ______ $\qquad$ $y = \sin 45° =$ ______

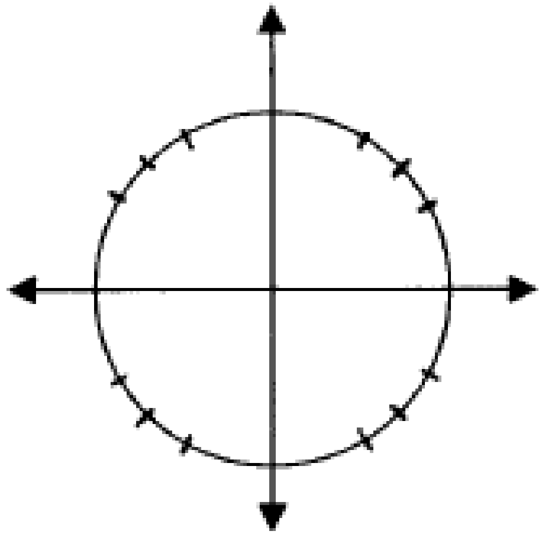
d) Complete the diagram by writing the coordinates of all points on the terminal side of the rays that use $\theta' = 45°$ as their reference. Recall the correspondence $(x, y) \leftrightarrow (\cos\theta, \sin\theta)$.



18) **Angles with Reference Angle 30° -** follow the same as above for all angles that use the 30° angle as a reference. Write down the values of their x- and y- coordinates.



19) **Angles with Reference Angle 60˚ -** same as above. Complete the diagram.



Ready for Problem Set 2 (17-25)

PROBLEM SET 1 (1-16)

State whether each expression is positive or negative:

1) $\sin 150^\circ$ _____ 2) $\cos 235^\circ$ _____ 3) $\sin(-325^\circ)$ _____

4) $\sin \frac{5\pi}{6}$ _____ 5) $\cos \frac{2\pi}{3}$ _____ 6) $\sin 4$ _____

7) Does $\sin\theta$ increase or decrease as:
   a) θ increases from 0˚ to 90˚ ______
   b) θ increases from 90˚ to 180˚? ______
   c) θ increases from 180˚ to 270˚? ______
   d) θ increases from 270˚ to 360˚? ______

Without using a calculator, solve each equation for all θ in radians.
8) $\cos\theta = -1$ 9) $\sin\theta = -1$

Items 10-12: Make a sketch.

10) If $\cos\theta = \frac{4}{5}$ and θ is in Quadrant I, what is $\sin\theta$? ________

11) If $\sin\theta = -\frac{24}{25}$ and θ is in Quad III, what is $\cos\theta$? ________

12) If $\cos\theta = -\frac{5}{13}$ and θ is in Quad II, what is $\sin\theta$? ________

Without using a calculator, complete with one of the symbols <, >, =.

13) $\sin 50°$ _____ $\sin 30°$

14) $\cos 45°$ _____ $\cos 30°$

15) $\sin 188°$ _____ $\sin 8°$

16) $\cos 50°$ _____ $\cos(-50°)$

PROBLEM SET 2: (17-25)

17) Express each of the following in terms of the sine or cosine of a reference angle. The first is given as an example. When no degree (°) symbol is present, assume the angle is in radians.

a) $\sin 138° = \sin 42°$ reason: reference angle for θ=138° is θ'=42° and θ is in Quad II, hence positive.

b) $\cos 138° =$ ______

c) $\sin(-37°) =$ ______

d) $\cos 834° =$ ______

e) $\cos(-132°) =$ ______

Items 18- :Study the sine and cosine values of 30°,45°,and 60°. Then give the exact value of each expression in simplest radical form.

18) sin 225°

19) cos 300°

20) $\sin\left(-\frac{\pi}{3}\right)$

21) $\cos\left(\frac{7\pi}{6}\right)$

22) $\cos\frac{3\pi}{2}$

23) $\sin\left(-\frac{3\pi}{2}\right)$

24) Imagine a particle starting at (1,0) and making one counterclockwise revolution on the unit circle. Let θ be the angle in standard position that corresponds to the particle's position.

a) At how many points along the path of the particle are the x- and y-coordinates equal? ____

b) What values of θ correspond to the points in part (a)? _________

25) Complete the unit circle below with all x- and y-coordinates of points on the terminal side of all quadrantal angles as well as angles that use 30°, 45°, and 60° as a reference.